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Unspanned Stochastic Volatility and Random Field Models

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Foreword

There is still a consistency problem if we want to price interest rate derivatives on zero bonds, like caplets or floorlets, and on swaps, like swaptions, at the same time within one model. The popular market models concentrate either on the valuation of caps and floors or on swaptions, respectively. Musiela and Rutkowski (2005) put it this way: ”We conclude that lognormal market models of forward LIBORs and forward swap rates are inherently inconsistent with each other. A challenging practical question of the choice of a benchmark model for simultaneous pricing and hedging of LIBOR and swap derivatives thus arises.”

Repplinger contributes to the research in this area with a new systematic approach. He develops a generalized Edgeworth expansion technique, called Integrated Edgeworth Expansion (IEE), to overcome the aforementioned consistency problem. Together with a ’state of the art’ Fractional Fourier Transform technique (FRFT) for the pricing of caps and floors this method is applied to price swaptions within a set of ’up to date’ multidimensional stochastic interest rate models. Beside the traditional multi-factor Heath-Jarrow-Morton models, term structure models driven by random fields and models with unspanned stochastic volatility are successfully covered. Along the way some new closed form solutions are presented.

I am rather impressed by the results of this thesis and I am sure, that this monograph will be most useful for researchers and practitioners in the field.

Tübingen, May 2008 *Rainer Schöbel*

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Laupen, Switzerland, May 2008 *Detlef Repplinger*

Contents

Chapter 1
Introduction

Before the work of Ho and Lee [39] and Heath, Jarrow and Morton [35] the point of view in the literature was explaining the term structure of interest rates or respectively the cross section of bond prices. The new Heath, Jarrow and Morton (HJM) models perfectly fit to an observed initial term structure by focussing on the arbitrage-free pricing of related derivatives. Given a specification of the volatility for the forward rates or bond prices together with the initial term structure completely determines the risk-neutral bond price dynamics or equivalently the short rate process (see e.g. de Jong and Santa Clara [24], Casassus, Collin-Dufresne and Goldstein [14]). The volatility structure in general can be computed by inverting the option prices similar to the calculation of implied volatilities that are extracted from stock option prices. One drawback of these models lies in the non-Markovian structure of the short rate dynamics resulting in a computationally low tractability. Hence, most of the HJM-models in the literature are restricted to a deterministic volatility structure leading to a Markovian short rate process. It is well known that a deterministic volatility function always leads to Gaussian interest rates and therefore we have to deal with negative interest rates.

Unlike these models the traditional models, such as Cox, Ingersoll, and Ross [22] and Vasicek [73] are built on state variables starting from a given short rate process. Hence, they directly causes a Markovian structure. On the other hand it is well known that they can fit the initial term structure only by making the model parameters time dependent. In contrary, coming from the HJM-framework the so called extended form models (e.g. Hull and White [41]) are a result of the arbitrage-free HJM setting, where the short rate dynamics are defined endogenously.

Cox, Ingersoll and Ross [22] and Jamshidian [42] demonstrate that closed-form solutions for zero-coupon bond options can be derived for single-factor square root and Gaussian models. More generally, Duffie, Pan and Singleton

[28] demonstrate that the entire class of affine models possesses closed-form solutions for zero-coupon bond options, which can be derived by applying standard Fourier inversion techniques. The option pricing formula for zero-coupon bond options (caplets/floorlets) are discussed e.g. in Chen and Scott [16], Duffie, Pan and Singleton [28], Bakshi and Madan [6] and Chacko and Das [15]. Unfortunately, these papers say only little about the pricing of options on coupon bonds (swaptions).

Given a single-factor Gaussian interest rate model Jamshidian [42] derives a closed-form solution for the price of an option on coupon bonds. This solution stems from the fact that the optimal exercise decision at maturity is a one dimensional boundary and a coupon bond can be written as a portfolio of zero-coupon bonds. Unfortunately, the closed-form solution for options on coupon bonds and zero-coupon bond options cannot be extended to multi-factor models. Then the exercise boundary becomes a non-linear function of the multiple state variables and cannot be computed in closed-form. Litterman and Scheinkman [55] shows that typically multi-factor models are needed to capture the dynamics of the term structure of interest rates. One approach to handle the non-linear exercise boundary in a multi-factor setting is applied by Singleton and Umantsev [68]. They approximate the exercise boundary with a linear function of the multiple state-variables. On the other hand, there are two new drawbacks from this approach. Firstly, a separate approximation has to be performed for every single strike price coming along with a low efficiency and tractability. Secondly, and even more restrictive their approach becomes completely intractable for a large number of state variables. Hence, we need a new method for the computation of bond option prices in a generalized multi-factor HJM setup.

Miltersen, Sandmann, and Sondermann [60] and Brace, Gatarek, and Musiela [9] derive the so called LIBOR market model. In their approach the lognormal distributed interest rates are given and closed-form solutions can be derived to compute the prices of interest rate caps/floors and swaptions. Their formulae are very tractable and easy to handle. On the other hand, there occurs a model inconsistency between the swaption and cap/floor markets coming from the fact that a lognormal LIBOR rate cannot coexist with a lognormal distributed swap rate.

We overcome this inconsistency, by deriving a unified framework that directly leads to consistent cap/floor and swaption prices. Thus, in general we start from a HJM-like framework. This framework includes the traditional HJM model as well as an extended approach, where the forward rates are driven by multiple Random Fields. Furthermore, even in the case of a multi-factor unspanned stochastic volatility (USV) model we are able to compute the bond option prices very accurately. First, we make an exponential affine guess for the solution of an expectation, which is comparable to the solution of a special characteristic function. Then, given this solution we are able to compute the prices of zero-coupon bond options by applying standard Fourier inversion techniques. In limited cases this method can also be applied for the pricing of coupon bearing bond options (see e.g. Singleton and Umantsev [68]), but completely fails assuming a multi-factor framework. In order to overcome this drawback, we use the solution of our exponential affine guess to compute the moments of the underlying random variable. Given these moments we are then able to compute the prices of coupon bond options (swaptions) by performing an integrated approach of a generalized Edgeworth Expansion (IEE) technique (see chapter (4)). This is a new method for the computation of the probability that an option matures in the money.

In chapter (2), we derive a unified framework for the computation of the price of an option on a zero-coupon bond and a coupon bond by applying the well known Fourier inversion scheme. Therefore, we introduce the transform $\Theta_t(z)$, which later on can be seen as a characteristic function. In case of zero-coupon bond options we are able to find a closed-form solution for the transform $\Theta_t(z)$ and apply standard Fourier inversion techniques. Unfortunately, assuming a multi-factor framework there exists no closed-form solution of the characteristic function $\Xi_t(z)$ given a coupon bond option. Hence, in this case Fourier inversion techniques fail.

Chapter (3) focuses on the derivation of a generalized approach of the Edgeworth Expansion (EE) technique. This approach extends the series expansion technique of Jarrow and Rudd [44], Turnbull and Wakeman [72], Collin-Dufresne and Goldstein [19] and Ju [47] to a generalized approximation scheme for the computation of the exercise probabilities that an option ends up in the money. The main advantage of this new technique stems from the fact that the pricing scheme is strictly separated from the underlying model structure. Thus, the structure of the underlying dynamics enter only in the computation of the moments. In other words, we derive a generalized algorithm to approximate the exercise probabilities, by using only the moments of the underlying random variable, which either can be computed in closed-form or even numerically[1].

In chapter (4), we derive a new integrated version of the generalized EE[2]. This integrated version can be applied to compute the exercise probabilities directly, instead of computing an integration over the approximated pdf[3]. Finally, we obtain a series expansion of the exercise probabilities in terms of Hermite polynomials and cumulants. This approach is a technique to approx-

[1] A Matlab program is available in the appendix section (10.4).

[2] Therefore, we term this generalized series expansion the Integrated Edgeworth Expansion (IEE).

[3] The EE originally is derived to approximate density functions instead of probabilities.

imate the cumulative density function (cdf), even when there exists no solution for the characteristic function. Hence, the approach is a new generalized approximation scheme especially adapted for the use in option pricing theory, where we are interested in the computation of the exercise probabilities. Then, we show that the IEE approach is very accurate for the approximation of a χ^2_ν- and the lognormal-cdf. Furthermore, we show that the series expansion of a characteristic function can also be applied for lognormal distributed random variables. The divergence of the series expansion (Leipnik [53]) can be avoided by using only the terms up to a critical order M_c for which the series expansion converges. Thus, we conclude that the application of the new IEE is admissible for practical use and leads to excellent results for the price of fixed income derivatives, even if the underlying is lognormaly distributed.

In chapter (5), we start from a traditional Heath, Jarrow and Morton (HJM) approach and derive the pricing formulae of the aforementioned fixed income derivatives . Given the HJM [35] restrictions for the volatility function $\sigma(t,T)$, implying an arbitrage-free model structure ,we implicitly obtain the arbitrage-free bond price process or equivalently the corresponding short rate dynamics. Then, by solving a set of coupled ordinary differential equations (ODE), we obtain an exponential affine approach to compute the characteristic function in closed-form. Finally, the well known closed-form solution for the price of an option on a discount bond can be derived by calculating the Fourier inversion of the characteristic function. By the use of this closed-form solution, we introduce the Fractional Fourier Transform (FRFT) technique. Then the prices of zero-coupon bond options can be computed very efficiently for a wide range of strike prices by performing this advanced Fourier inversion method. Unfortunately, this technique cannot be applied for the computation of options on coupon bonds in a multi-factor framework. Hence, thereafter we apply the new IEE technique to compute the price of a coupon bearing bond option in a multi-factor HJM-framework.

In chapter (6), we extend the traditional HJM approach, by assuming that the sources of uncertainty are driven by Random Fields. For that reason, we introduce a non-differentiable Random Field (RF) and an equivalent T-differentiable counterpart. Given the particular Random Field, we derive the corresponding short rate model and show in contrast to Santa-Clara and Sornette [67] and Goldstein [33] that only a T-differentiable RF leads to admissible well-defined short rate dynamics[4]. Santa-Clara and Sornette [67] argue that there is no empirical evidence for a T-differentiable RF. We conclude that the existence of some pre-defined short rate dynamics enforces the usage of a T-differentiable RF. Furthermore, we compute bond option prices when

[4] In the sense that the derivative of the RF with respect to the term T is defined.

the term structure is driven by multiple Random Fields[5]. The higher option prices for ”out-of-the-money” options resulting from the RF term structure models could help to explain the implied volatility skew observed e.g. by Casassus, Collin-Dufresne, and Goldstein [14] and Li and Zhao [54].

Finally, we introduce a term structure model with unspanned stochastic volatility (USV) in chapter (7). Collin-Dufresne and Goldstein [18], Heiddari and Wu [36], and in more recent work Jarrow, Li, and Zhao [45] and Li, Zhao [54] show that the prices of swaptions and caps/floors appear to be driven by risk factors that do not effect the term structure. Hence, interest rate option markets exhibit risk factors unspanned by the underlying yield curve of interest rates. This directly implies that bond options cannot be replicated and hedged perfectly by trading solely bonds. As a result the bond markets do not span the fixed income derivative markets and these markets becomes incomplete. We introduce a general multi-factor HJM- framework with USV combined with correlated sources of uncertainty[6]. Then, by applying the FRFT- or IEE-technique, together with our new solution for correlated sources of uncertainty, we are able to compute the prices of bond prices very efficiently and accurately. Note that our approach remains tractable and accurate, even in the case of a multi-factor framework combined with USV. The higher prices we obtain for ”out-of-the-money” options indicate that a dependency structure between the forward rate dynamics and the stochastic volatility process could help to explain the implied volatility smile observed in the LIBOR-based fixed income derivative markets (see e.g. Casassus, Collin-Dufresne, and Goldstein [14] and Li and Zhao [54]).

In chapter (8) we review and conclude our results and give ideas for further extensions of this work.

[5] An example of a two-factor RF model could e.g. be enforced by a separate modeling of bond prices for corporate bonds and default spreads.

[6] Han 2007 [34] showed in an empirical analysis assuming a similar model, but excluding a potential correlation between the forward rate process and the subordinated stochastic volatility process, that the average relative pricing error between the cap markets and the no-arbitrage values implied by the swaption markets are in the range of the bid-ask spread. Nevertheless, the average absolute relative pricing error can even exceed 6% in his study.

Chapter 2
The option pricing framework

The option markets based on swap rates or the LIBOR have become the largest fixed income markets, and caps (floors) and swaptions are the most important derivatives within these markets. Thereby, a cap (floor) can be interpreted as a portfolio of options on zero bonds. Hence, pricing a cap (floor) is very easy, if we have found an exact solution for the arbitrage-free price of a caplet (floorlet) (see e.g. Briys, Crouhy and Schöbel [11]. On the other hand, a swaption may be interpreted as an option on a portfolio of zero bonds[1]. Therefore, even in the simplest case of lognormal-distributed bond prices, the portfolio of the bonds would be described by the distribution of a sum of lognormal-distributed random variables. Unfortunately, there exists no analytic density function for such a sum of lognormal-distributed random variables. Hence, using a multi-factor model with Brownian motions or Random Fields[2] as the sources of uncertainty, it seems unlikely that exact closed-form solutions can be found for the pricing of swaptions. The characteristic function of the random variable $\bar{X}(T_{0,}\{T_i\}) = \log\sum_{i=1}^{u} c_i P(T_0, T_i)$ with the coupon payments c_i at the fixed dates $T_i \in \{T_{1,}..., T_u\}$ cannot be computed in closed-form. Otherwise, we are able to find a closed-form solution for the moments of the underlying random variable $V(T_{0,}\{T_i\}) = \sum_{i=1}^{u} c_i P(T_0, T_i)$ at the exercise date T_0 of the swaption. Hence, using the analytic solution of the moments within our Integrated Edgeworth Expansion (IEE) enables us to compute the T_i-forward measure exercise probabilities $\Pi_t^{T_i}[K] = E_t^{T_i}\left[\mathbf{1}_{V(T_0,\{T_i\})>K}\right]$ (section (5.3.3)). Reasonable carefulness has to be paid for the fact that the characteristic function of a lognormal-distributed

[1] The owner of a swaption with strike price K maturing at time T_0, has the right to enter at time T_0 the underlying forward swap settled in arreas. A swaption may also be seen as an option on a coupon bearing bond (see e.g. Musiela and Rutkowski [61]).

[2] Eberlein and Kluge [29] find a closed-form solution for swaptions using a Lévy term structure model. A solution for bond options assuming a one-factor model has been derived by Jamishidian [42].

random variable cannot be approximated asymptotically by an infinite Taylor series expansion of the moments (Leipnik [53]). As a result of the Leipnik-effect we truncate the Taylor series before the expansion of the characteristic function tends to diverge.

In contrary to the computation of options on coupon bearing bonds via an IEE, we can apply standard Fourier inversion techniques for the derivation zero bond option prices. Applying e.g. the Fractional Fourier Transform (FRFT) technique of Bailey and Swarztrauber [4] is a very efficient method to compute option prices for a wide range of strike prices. This can either be done, by directly computing the option price via an Fourier inversion of the transformed payoff function or by separately computing the exercise probabilities $\Pi_t^{T_i}[k]$. Running the first approach has the advantage that we only have to compute one integral for the computation of the option prices. On the other hand, sometimes we are additionally interested in the computation of single exercise probabilities[3]. Therefore, we prefer the latter as the option price can be easily computed by summing over the single probabilities[4].

2.1 Zero-coupon bond options

In the following, we derive a theoretical pricing framework for the computation of options on bond applying standard Fourier inversion techniques. Starting with a plain vanilla European option on a zero-coupon bond with the strike price K, maturity T_1 of the underlying bond and exercise date T_0 of the option, we have

$$ZBO_{\mathrm{w}}(t,T_0,T_1) = \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\left(P(T_0,T_1)-K\right)\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right] \quad (2.1)$$

$$= \mathrm{w}E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+X(T_0,T_1)}\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right] - \mathrm{w}KE_t^Q\left[e^{-\int_t^{T_0} r(s)ds}\mathbf{1}_{\mathrm{w}X(T_0,T_1)>\mathrm{w}k}\right],$$

with $\mathrm{w}=1$ for a European call option and $\mathrm{w}=-1$ for a European put option[5]. We define the probability $\Pi_{t,a}^Q[k]$ given by

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[3] Note that the FRFT approach is very efficient. Hence, the computation of single exercise probabilities runs nearly without any additional computational costs and without getting an significant increase in the approximation error (see e.g. figure (5.1)).

[4] Furthermore, we want to be consistent with our IEE approach, where the price of the coupon-bond options can only be computed by summing over the single exercise probabilities $\Pi_t^{T_i}[K]$

[5] In this thesis, we mainly focus on the derivation of call options (w = 1), keeping in mind that it is always easy to compute the appropriate probabilities for w = -1 via $E_t^Q\left[e^{-\int_t^{T_0} r(s)ds+aX(T_0,T_1)}\mathbf{1}_{X(T_0,T_1)<k}\right] = 1-\Pi_{t,a}^Q[k]$.